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Algebras for Flavor Quantum Numbers

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It is shown that the flavor quantum numbers of the basic elementary particles, leptons and quarks, as well as hadrons (with quarks as constituents), can be described with $SU(2)\times U(1)$ type of algebras. To treat simultaneously leptons and quarks (hadrons), we introduce the grace quantum number, G, in place of L (the total lepton quantum number) and B (the baryon quantum number). The formalism developed here requires the basic elementary particles to come in even numbers. For the case of four basic particles we have quantum numbers denoted as Q, X, and Y and their duals denoted as Q', X', and Y'. For the four leptons Q is the ordinary charge, while $-Y$ and Y' are L_μ (the muon lepton number) and L_e (the electron lepton number), respectively. For the four quarks Q is the ordinary charge , Y the ordinary hypercharge, while X, a new quantum number, is simply the X charge, which, however, can be related to charm C.

1. INTRODUCTION

In this article we wish to show that the flavor quantum numbers of the basic elementary particles, quarks and leptons, can be associated with the $SU(2)\times U(1)$ type of algebras. Furthermore, if it is required that the basic particles belong to the fundamental, **2**, representations of corresponding $SU(2)$'s, then the number of respective particles must be even.

In this formalism the quarks are distinguished from leptons by the grace quantum number, G, which for quarks (hadrons) is B (the baryon quantum number) and for leptons is L (the total lepton quantum number).

In Section 2 some general properties of an $SU(2)\times U(1)$ algebra is discussed, where the grace quantum number, G, is introduced as the generator of $U(1)$.

Section 3 deals specifically with four leptons and four quarks, respectively.

Discussion and summary are given in Section 4. Here some of the properties of the quark and lepton quantum numbers for various interactions are briefly discussed.

2. GENERALITIES

We shall assume that the basic elementary particles are spin-1/2 fermions. If there are n of such particles, then we further assume that they can be classified according to the fundamental, **n**, representation of some $SU(n)$, which is denoted as

$$\psi(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \vdots\\ \psi_n(x)\end{pmatrix} \tag{1}$$

Of course this does not imply that the masses of these particles are degenerate or that their interactions are necessarily invariant under $SU(n)$ transformations. $SU(n)$ generators ${}^aA_b$ $(a, b=1,\ldots, n)$ satisfy

$$\begin{gathered}[{}^aA_b, {}^cA_d]=\delta_b^c\,{}^aA_d-\delta_d^a\,{}^cA_b\\ {}^aA_b^\dagger={}^bA_a, \qquad \sum_a {}^aA_a=0\end{gathered} \tag{2}$$

In terms of $U(n)$ generators ${}^a\tilde{A}_b$, ${}^aA_b$ are given as

$${}^aA_b={}^a\tilde{A}_b-\frac{1}{n}\delta_b^a\sum_c {}^c\tilde{A}_c \tag{3}$$

One easily verifies that $\tilde{A}$'s satisfy the same commutation rules as A's do in relation (2).

The transformation properties of $\psi(x)$ and $\psi^\dagger(x)$ under $SU(n)$ are determined with

$$\begin{gathered}[{}^aA_b, \psi(x)]=-{}^a\alpha_b\psi(x)\\ [{}^aA_b, \psi^\dagger(x)]=\psi^\dagger(x)\,{}^a\alpha_b\end{gathered} \tag{4}$$

with $n \times n$ matrices ${}^a\alpha_b$ (${}^a\alpha_b^\dagger = {}^b\alpha_a$) given as

$$({}^a\alpha_b)_i^j = \delta_i^a \delta_b^j - (1/n)\delta_i^j \delta_b^a$$

$$\sum_a {}^a\alpha_a = 0 \tag{5}$$

Similarly, ${}^a\tilde{A}_b$ will determine the transformation properties of $\psi(x)$ and $\psi^\dagger(x)$ under $U(n)$ where, instead of ${}^a\alpha_b$ in (4), we now have $n \times n$ matrices ${}^a\tilde{\alpha}_b$ (${}^a\tilde{\alpha}_b^\dagger = {}^b\tilde{\alpha}_a$) given as

$$({}^a\tilde{\alpha}_b)_i^j = \delta_i^a \delta_b^j$$

$$\sum_a {}^a\tilde{\alpha}_a = I(n) \tag{6}$$

Here $I(n)$ is the $n \times n$ unit matrix.

Relations (4) are equal time commutation relations (ETC) and in these specific cases ${}^aA_b$ can be constructed from currents

$${}^aj_b^\mu(x) = \bar{\psi}(x)\gamma^\mu {}^a\alpha_b \psi(x) \tag{7}$$

as

$${}^aA_b = \int d^3x\, {}^aj_b^4(x) \tag{8}$$

Replacing in (8) ${}^a\alpha_b$ with ${}^a\tilde{\alpha}_b$, we get

$${}^a\tilde{j}_b^\mu(x) = \bar{\psi}(x)\gamma^\mu {}^a\tilde{\alpha}_b \psi(x) \tag{9}$$

from which ${}^a\tilde{A}_b$ can be defined in a similar manner.

Algebra (2) is assumed to be valid generally, i.e., also for particles that are composites of basic particles described with $\psi(x)$. In this case ${}^aA_b$ can be given in terms of field operators of composite particles rather than $\psi(x)$.

With the generators of $SU(n)$ algebra, we can get $n(n-1)/2$ $SU(2)$ algebras as follows:

$$\begin{aligned}
[{}^aA_b, {}^bA_a] &= {}^aA_a - {}^bA_b \\
[{}^aA_a - {}^bA_b, {}^aA_b] &= 2\,{}^aA_b \\
[{}^aA_a - {}^bA_b, {}^bA_a] &= -2\,{}^bA_a \\
a<b, &\qquad a, b = 1, 2, \ldots, n
\end{aligned} \tag{10}$$

These $SU(2)$ algebras are not yet those from which to deduce the quantum numbers. Rather, we define (with ϕ_i real)

$$H_+ = \sum_{i=1}^{r} e^{i\phi_i\, a_i}A_{b_i}$$

$$\begin{aligned} &a_i \neq a_j\ (i\neq j), \qquad b_i \neq b_j\ (i\neq j) \\ &a_i \neq b_j, \qquad a_i, b_j = 1,2,\ldots,n \\ &i, j = 1,2,\ldots,r \\ &2r = \begin{cases} (n-1), & n \text{ odd} \\ n, & n \text{ even} \end{cases} \end{aligned} \tag{11a}$$

where clearly

$$H_- = H_+^\dagger$$

$$H_3 = \frac{1}{2}\sum_{i=1}^{r}\left({}^{a_i}A_{a_i} - {}^{b_i}A_{b_i}\right) \tag{11b}$$

Thus

$$\begin{aligned} [H_+, H_-] &= 2H_3 \\ [H_3, H_\pm] &= \pm H_\pm \end{aligned} \tag{12}$$

which is an $SU(2)$ algebra. In general there will be more than one such $SU(2)$ algebra if $n>2$. However, two such $SU(2)$ algebras with generators $H_\pm$, H_3 and $H'_\pm$, H'_3 are equivalent if

$$H'_3 = \pm H_3 \tag{13}$$

Now, demanding that the basic particles belong to the fundamental, **2**, representation of $SU(2)$ from (12), with the stipulation that no component of $\psi(x)$ from (1) remains invariant under the action of all H's from (12), then n, the number of basic particles described by $\psi(x)$, must be even. To see this let us look at

$$[H_3, \psi(x)] = -h_3\psi(x) \tag{14}$$

where diagonal $n\times n$ matrix h_3 is given as [compare with (11b)]

$$h_3=\frac{1}{2}\sum_{i=1}^{r}\left({}^{a_i}\alpha_{a_i}-{}^{b_i}\alpha_{b_i}\right) \tag{15}$$

From (5) we see that h_3 will be a diagonal matrix with values $1/2$, say m times, and with values $(-1/2)$, say $(n-m)$ times. Since

$$\mathrm{Tr}\, h_3=0=m(1/2)+(n-m)(-1/2)$$

we must have

$$n=2m \tag{16}$$

which proves the assertion.

With $n=2m$ $(r=m)$, the number N_{2m} of $SU(2)$ algebras (12) is

$$N_{2m}=\frac{1}{2}\frac{(2m)!}{m!} \tag{17}$$

giving

$$N_2=1, \qquad N_4=6, \qquad N_6=60, \qquad \text{etc.} \tag{18}$$

As we see, the number of $SU(2)$ algebras (12) increases rather rapidly with the increase in the number of basic elementary particles.

Next we define the grace, G, quantum number as

$$G=g\sum_{a=1}^{2m}{}^{a}\tilde{A}_a \tag{19}$$

When $g=1$, we are talking about leptons ($\psi_1=e, \psi_2=\nu_e$, etc.). Now $G=L$, the total lepton quantum number. When $g=1/3$. we are talking about quarks ($\psi_1=u, \psi_2=d$, etc.), and $G=B$, the baryon quantum number.

Now for a particular $SU(2)$ algebra (12) we define a corresponding quantum number Q_H and its dual Q'_H as

$$\tfrac{1}{2}G=Q_H-H_3=H_3-Q'_H \tag{20}$$

If Q_H and Q'_H are to be quantum numbers associated with algebra (12), then the eigenstates of H_3 should be also the eigenstates of Q_H and Q'_H.

Consequently

$$[G, H_{\pm}]=[G, H_3]=0 \tag{21}$$

that is, the operators H_+, H_-, and H_3 are the generators of a $SU(2)$ group, and $G/2$ is the generator of an $U(1)$ group; together these form an $SU(2)\times U(1)$ group.

Relation (20) actually consists of two equations, which can be cast as

$$G=Q_H-Q'_H \tag{22}$$

$$2H_3=Q_H+Q'_H \tag{23}$$

Already from (22) and (23) we can deduce some very interesting properties of Q_H and Q'_H. For example, since the eigenvalues of $2H_3$ are ± 1 no matter how many basic particles we have, the eigenvalues of Q_H (Q'_H) numerically can assume only two values. Specifically, from (22) and (23) we have (with q_H and q'_H denoting the eigenvalues of Q_H and Q'_H, respectively)

$$g=q_H-q'_H$$

$$\pm 1=q_H+q'_H$$

from which

$$q_H=\tfrac{1}{2}(g\pm 1) \tag{24a}$$

$$q'_H=\tfrac{1}{2}(-g\pm 1) \tag{24b}$$

Once g is specified, q_H and q'_H are specified completely.

Relations (24a) and (24b) will be particularly important when the roles of Q_H and Q'_H are reversed; i.e., when we call Q'_H a quantum number and Q_H its dual. As we shall see later, this will actually be the case with X and Y charges.

3. CASES OF FOUR BASIC PARTICLES

Here we shall apply the results from Section 2 to the case of four basic particles.

Let us now denote the generators of six $SU(2)$ subgroups of $SU(4)$ as follows:

$$I_+={}^1A_2, \qquad I_-={}^2A_1, \qquad I_3=\tfrac{1}{2}({}^1A_1-{}^2A_2) \tag{25a}$$

$$V_+={}^1A_3, \qquad V_-={}^3A_1, \qquad V_3=\tfrac{1}{2}({}^1A_1-{}^3A_3) \tag{25b}$$

$$U_+={}^2A_3, \qquad U_-={}^3A_2, \qquad U_3=\tfrac{1}{2}({}^2A_2-{}^3A_3) \tag{25c}$$

$$L_+ = {}^1A_4, \qquad L_- = {}^4A_1, \qquad L_3 = \tfrac{1}{2}\left({}^1A_1 - {}^4A_4\right) \tag{25d}$$

$$M_+ = {}^2A_4, \qquad M_- = {}^4A_2, \qquad M_3 = \tfrac{1}{2}\left({}^2A_2 - {}^4A_4\right) \tag{25e}$$

$$K_+ = {}^3A_4, \qquad K_- = {}^4A_3, \qquad K_3 = \tfrac{1}{2}\left({}^3A_3 - {}^4A_4\right) \tag{25f}$$

The notation used here is an extension of I, V, and U spin notations introduced by Meshkov, Levinson, and Lipkin (1963) within the context of $SU(3)$. Following general formulas (11a), (11b), (12), and (20), the $SU(2)\times U(1)$ generators with associated quantum number operators are then given as

$$\begin{gathered} Z_+ = e^{i\phi_I} I_+ + e^{i\phi_K} K_+ \\ Z_- = Z_+^\dagger, \qquad Z_3 = I_3 + K_3 \\ G/2 = Z_3 - X = X' - Z_3 \end{gathered} \tag{26}$$

$$\begin{gathered} R_+ = e^{i\phi_U} U_+ + e^{i\phi_L} L_+ \\ R_- = R_+^\dagger, \qquad R_3 = U_3 + L_3 \\ G/2 = R_3 - Y = Y' - R_3 \end{gathered} \tag{27}$$

$$\begin{gathered} W_+ = e^{i\phi_V} V_+ + e^{i\phi_M} M_+ \\ W_- = W_+^\dagger, \qquad W_3 = V_3 + M_3 \\ G/2 = W_3 - Y = Y' - W_3 \end{gathered} \tag{28}$$

$$\begin{gathered} \tilde{Z}_+ = e^{i\phi_I} I_+ + e^{-i\phi_K} K_- \\ \tilde{Z}_- = \tilde{Z}_+^\dagger, \qquad \tilde{Z}_3 = I_3 - K_3 \\ G/2 = Q - \tilde{Z}_3 = \tilde{Z}_3 - Q' \end{gathered} \tag{29}$$

$$\begin{gathered} \tilde{R}_+ = e^{i\phi_U} U_+ + e^{-i\phi_L} L_- \\ \tilde{R}_- = \tilde{R}_+^\dagger, \qquad \tilde{R}_3 = U_3 - L_3 \\ G/2 = -X - \tilde{R}_3 = \tilde{R}_3 + X' \end{gathered} \tag{30}$$

$$\begin{gathered} \tilde{W}_+ = e^{i\phi_V} V_+ + e^{-i\phi_M} M_- \\ \tilde{W}_- = \tilde{W}_+^\dagger, \qquad \tilde{W}_3 = V_3 - M_3 \\ G/2 = Q - \tilde{W}_3 = \tilde{W}_3 - Q' \end{gathered} \tag{31}$$

In relations (26) to (31), Q, X, and Y are the charge, X-charge, and hypercharge quantum number operators, with Q', X', and Y' being their duals. We notice that, despite six $SU(2)\times U(1)$ algebras, we have actually only three quantum numbers. This follows from the fact that

$$R_3 = W_3, \qquad Z_3 = -\tilde{R}_3, \qquad \tilde{W}_3 = \tilde{Z}_3 \tag{32}$$

as can be easily verified with the help of (25a)–(25f). Of course, because $SU(4)$ is of rank 3, we expect only three quantum numbers in addition to G.

Now, since

$$Q - \tilde{W}_3 - Z_3 + X = 0$$

we get

$$Q + X = Z_3 + \tilde{W}_3 = 2I_3$$

where relations (25) have been utilized. By similar manipulations, we can get more than just this relation, and we quote them all here:

$$\begin{aligned} Q+X &= 2I_3 \\ Q+Y &= 2V_3 \\ Y-X &= 2U_3 \end{aligned} \tag{33}$$

$$\begin{aligned} Q-X &= G-2K_3 \\ Q-Y &= G-2M_3 \\ Y+X &= -G+2L_3 \end{aligned} \tag{34}$$

Similar relations can be obtained for Q', X', and Y' by replacing G with $(-G)$ in the above relations.

3.1. Four Leptons. The field operators of four spin-1/2 leptons are arranged as

$$\psi(x) = \begin{pmatrix} e(x) \\ \nu_e(x) \\ \nu_\mu(x) \\ \mu(x) \end{pmatrix} \tag{35}$$

i.e., $\psi_1 = e$, etc. The important thing here is that since neutrinos appear only with negative (left-handed) helicities and antineutrinos only with positive (right-handed) helicities, the field operators $\nu_e(x)$ and $\nu_\mu(x)$ are left-handed field operators.

Now identifying the grace quantum number, G, with the total lepton quantum number [$g=1$ in relation (19)]

$$L = \sum_{a=1}^{4} {}^a\tilde{A}_a \tag{36}$$

we can readily deduce from relations (26)–(31) the eigenvalues of various quantum numbers. In Table I we give the eigenvalues that correspond to L, Q, X, Y, Q', X', and Y' for each of the four leptons: e, ν_e, ν_μ, and μ. One should notice that these eigenvalues are negative relative to the same eigenvalues for quarks (to be discussed subsequently). We notice from Table I that the electron and muon lepton number operators are given as

$$L_e = Y' \tag{37a}$$

$$L_\mu = -Y \tag{37b}$$

In view of (27) or (28), we have $Y' - Y = L$, which now reads

$$L = L_e + L_\mu \tag{38}$$

It is remarkable how $SU(2)\times U(1)$ algebras (27) and (28) define L_μ and L_e and relate them in a natural way to L. Experimentally, L_e and L_μ (Y' and Y) are separately conserved. Q, of course, is always conserved, and because $Q - Q' = L$, Q' is also conserved as far as leptons are concerned. What about X and X'? As far as leptons are concerned, X and X' are clearly not conserved. If they were that would mean that the net transitions $e \leftrightarrow \nu_\mu$ and $\mu \leftrightarrow \nu_e$ are allowed, which, however, would violate the L_e and L_μ conservations.

From $SU(2)\times U(1)$ algebras (29) and (31) we should be able to deduce the leptonic charge-changing current that is important in the usual weak interactions.

TABLE I. The Eigenvalues of Quantum Number Operators for the Case of Four Leptons

	L	Q	X	Y	Q'	X'	Y'
e	1	1	0	0	0	1	1
ν_e	1	0	-1	0	-1	0	1
ν_μ	1	0	0	-1	-1	1	0
μ	1	1	-1	-1	0	0	0

The charge-changing current that corresponds to $\tilde{Z}_+$ from (29) is

$$J^\mu(x; \tilde{Z}_+) = e^{i\phi_I}\bar{e}(x)\gamma^\mu \nu_e(x) + e^{-i\phi_K}\bar{\mu}(x)\gamma^\mu \nu_\mu(x) \tag{39}$$

while the charge-changing current that corresponds to $\tilde{W}_+$ from (31) is

$$J^\mu(x; \tilde{W}_+) = e^{i\phi_V}\bar{e}(x)\gamma^\mu \nu_\mu(x) + e^{-i\phi_M}\bar{\mu}(x)\gamma^\mu \nu_e(x) \tag{40}$$

One should note that since the ν_e and ν_μ fields in (39) and (40) are left-handed, the e and μ fields are effectively left-handed also; i.e., (30) and (40) are effectively $V-A$ currents. Consequently, currents (39) and (40) are good candidates for a charge-changing leptonic current necessary to describe the usual weak interactions.

To make a final choice, let us analyze the conservation of quantum numbers by these currents. L_e and L_μ (Y' and Y) are clearly conserved by $J^\mu(x; \tilde{Z}_+)$, while X and X' are not. On the other hand, $J^\mu(x; \tilde{W}_+)$, while conserving X and X', is violating the conservation of L_e and L_μ. Hence the charge-changing leptonic current entering into the usual weak interactions is

$$l^\mu(x; +) = J^\mu(x; \tilde{Z}_+)|_{\phi_I=\phi_K=0} = \bar{e}(x)\gamma^\mu \nu_e(x) + \bar{\mu}(x)\gamma^\mu \nu_\mu(x) \tag{41}$$

since experimentally L_e and L_μ are conserved. In (41) we have set the phases equal to zero, which is equivalent to absorbing them into the field operators.

It is interesting to note that current (41) makes X and X' "maximally" nonconserved by weak interactions in which leptons participate.

3.2. Four Quarks. Here the field operators of four spin-1/2 quarks are arranged as

$$\psi(x) = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \\ c(x) \end{pmatrix} \tag{42}$$

that is $\psi_1 = u$, $\psi_2 = d$, etc. Here the grace quantum number, G, has to be identified with the baryon quantum number [$g = 1/3$ in relation (19)]

$$B = \frac{1}{3} \sum_{a=1}^{4} {}^a\tilde{A}_a \tag{43}$$

As in the case of leptons, from relations (26)–(31) we can easily get eigenvalues for Q, X, Y, Q', X', and Y'. Before we do that let us define the charm, C, and the strangeness, S, quantum numbers. The charm quantum number, C, we define as

$$C=\tfrac{1}{2}(Q-X-Y) \tag{44}$$

Utilizing (33) and (34), or simply (26)–(31) (with G replaced by B), we get

$$\begin{aligned} C&=\tfrac{1}{2}\left(\tfrac{3}{2}B+\tilde{Z}_3-Z_3-W_3\right) \\ &=\tfrac{3}{4}B+\tfrac{1}{2}(M_3-V_3)-K_3={}^4\tilde{A}_4 \end{aligned} \tag{45}$$

On the other hand, from the first relation in (33) we have $X=2I_3-Q$, which when substituted into (44) gives

$$Q=C+I_3+Y/2 \tag{46}$$

which is a familiar relation.

In a similar manner we can define strangeness quantum number S as

$$S=\tfrac{1}{2}(Q+Y-X)-B \tag{47}$$

Looking at (44) we see that

$$S=C+Y-B \tag{48}$$

Working out (48) we get

$$S=-{}^3\tilde{A}_3 \tag{49}$$

which is a familiar definition of S.

In Table II we give the eigenvalues that correspond to B, Q, X, Y, Q', X', Y', S, and C for each of the quarks, u (up), d (down), s (strange), and

TABLE II. The Eigenvalues of Quantum Number Operators for the Case of Four Quarks

	B	Q	X	Y	C	S	Q'	X'	Y'
u	1/3	2/3	1/3	1/3	0	0	1/3	2/3	2/3
d	1/3	−1/3	−2/3	1/3	0	0	−2/3	−1/3	2/3
s	1/3	−1/3	1/3	−2/3	0	−1	−2/3	2/3	−1/3
c	1/3	2/3	−2/3	−2/3	1	0	1/3	−1/3	−1/3

c (charmed). The eigenvalues for Q, X, etc., are negative relative to the corresponding eigenvalues for leptons from Table I.

Let us point out, however, that if one chooses B, Q, X, and Y as quantum number operators, then S and C are redundant, as one sees from relations (44) and (47).

As we know from relations (26)–(31), to each quantum number there corresponds two $SU(2)\times U(1)$ algebras. Specifically, for Q or Q' we have (29) and (31). Let us try to "unify" (29) and (31) into one $SU(2)\times U(1)$ algebra. First we define

$$F_+=\cos\theta_c\tilde{Z}_+ +\sin\theta_c\tilde{W}_+$$

$$F_-=F_+^\dagger \tag{50a}$$

where the rotation angle θ_c is identified with the Cabibbo angle. Now in order that

$$[F_+,F_-]=2F_3, \qquad F_3=\tilde{Z}_3=\tilde{W}_3 \tag{50b}$$

$$B/2=Q-F_3=F_3-Q' \tag{50c}$$

we must have

$$[\tilde{Z}_+,\tilde{W}_-]+[\tilde{W}_+,\tilde{Z}_-]=0 \tag{51}$$

It is easily seen that (51) is satisfied if

$$\phi_I+\phi_M=\pi+\phi_V+\phi_K \tag{52}$$

The vector current that corresponds to the F_- generator is

$$J^\mu(x;F_-)=\cos\theta_c\left[e^{-i\phi_I}\bar{d}(x)\gamma^\mu u(x)+e^{i\phi_K}\bar{s}(x)\gamma^\mu c(x)\right]$$

$$+\sin\theta_c\left[e^{-i\phi_V}\bar{s}(x)\gamma^\mu u(x)+e^{i\phi_M}\bar{d}(x)\gamma^\mu c(x)\right] \tag{53}$$

Invoking constraint (52), relation (53) can be understood in terms of the Cabibbo transformed fields, either

$$u_c(x)=e^{-i\phi_I}\left[\cos\theta_c u(x)-\sin\theta_c e^{i(\phi_V+\phi_K)}c(x)\right]$$

$$c_c(x)=e^{-i\phi_V}\left[\cos\theta_c e^{i(\phi_V+\phi_K)}c(x)+\sin\theta_c u(x)\right] \tag{54a}$$

or

$$s_c(x)=e^{-i\phi_K}\left[\cos\theta_c s(x)-\sin\theta_c e^{i(\phi_I-\phi_V)}d(x)\right]$$

$$d_c(x)=e^{i\phi_V}\left[\cos\theta_c e^{i(\phi_I-\phi_V)}d(x)+\sin\theta_c s(x)\right] \tag{54b}$$

where now

$$J^{\mu}(x;F_-)=\bar{d}_c(x)\gamma^{\mu}u(x)+\bar{s}_c(x)\gamma^{\mu}c(x)$$

$$=\bar{d}(x)\gamma^{\mu}u_c(x)+\bar{s}(x)\gamma^{\mu}c_c(x) \tag{53'}$$

Now all the phase factors in (54a) and (54b) can be absorbed into the field operators, which is equivalent to setting

$$\phi_I=\phi_V=\phi_K=0, \qquad \phi_M=\pi \tag{55}$$

Current $J^{\mu}(x;F_-)$, however, is not yet the charge-changing quark (hadron) weak current since it does not have a built-in *V-A* structure, which for leptons was automatic because of the masslessness of neutrinos. Hence, in order to get the charge-changing quark (hadron) current relevant in usual weak interactions, we must introduce the *V-A* structure by hand. This simply amounts to changing γ^{μ} into $\gamma^{\mu}\frac{1}{2}(1+\gamma_5)$ in (53′) giving

$$h^{\mu}(x;-)=\bar{d}_c(x)\tfrac{1}{2}\gamma^{\mu}(1+\gamma_5)u(x)+\bar{s}_c(x)\tfrac{1}{2}\gamma^{\mu}(1+\gamma_5)c(x)$$

$$=\bar{d}(x)\tfrac{1}{2}\gamma^{\mu}(1+\gamma_5)u_c(x)+\bar{s}(x)\tfrac{1}{2}\gamma^{\mu}(1+\gamma_5)c(x) \tag{56}$$

Finally we write down the equal time commutators between $h^{\mu}(x;-)$ and Q, X, and Y, respectively:

$$\left[Q,h^{\mu}(x;-)\right]=-h^{\mu}(x;-)$$

$$\left[X,h^{\mu}(x;-)\right]=\cos\theta_c\left[\bar{s}(x)\gamma^{\mu}\tfrac{1}{2}(1+\gamma_5)c(x)-\bar{d}(x)\gamma^{\mu}\tfrac{1}{2}(1+\gamma_5)u(x)\right]$$

$$\left[Y,h^{\mu}(x;-)\right]=-\sin\theta_c\left[\bar{d}(x)\gamma^{\mu}\tfrac{1}{2}(1+\gamma_5)c(x)+\bar{s}(x)\gamma^{\mu}\tfrac{1}{2}(1+\gamma_5)u(x)\right]$$

As the interaction Lagrangian density for the usual weak interactions of quarks is proportional to $h^{\dagger}_{\mu}(x;-)h^{\mu}(x;-)$, it is immediately evident that while Q is conserved, X and Y will not be conserved, $\cos\theta_c$ and $\sin\theta_c$ being the "strengths" of nonconservation, respectively.

4. DISCUSSION AND SUMMARY

We have shown that $SU(2)\times U(1)$ algebras yield all known quantum numbers of quarks and leptons. In the course of our discussion we have encountered a new quantum number, the X charge, which, however, for the case of four quarks can be related to the charm quantum number, C. For four leptons the Y and Y' quantum numbers were shown to be equivalent with $(-L_\mu)$ and L_e, respectively. The X and X' charges in the case of leptons are new quantum numbers, and as far as the charged leptonic current processes are concerned (weak interactions), they are "maximally" nonconserved.

Of course, rather than talking about the Q, X, and Y quantum number operators separately for leptons and quarks, we can define the "total" quantum number operators as (with q standing for "quark" ("hadron") and l standing for lepton)

$$\begin{aligned} Q_T &= Q_q - Q_l \\ X_T &= X_q - X_l \\ Y_T &= Y_q - Y_l \end{aligned} \tag{57}$$

Here we took into account that the eigenvalues of Q_l, X_l, and Y_l are in negative units with respect to the eigenvalues of Q_q, X_q, and Y_q, respectively.

One thing that we can say immediately is that Q_T must be conserved in all interactions. The quantum numbers X_T and Y_T, however, are not universally conserved. Specifically, while Y_l (Y_l') should always be conserved, the same is not true for Y_q (Y_q'). Nevertheless, the total quantum number operators as defined by (57) might be very useful in classifications of reactions. For example, let us take the following (weak interaction) process:

$$e^- + P \to \nu_e + N \tag{58}$$

One easily deduces that reaction (58) conserves Q_T, X_T, and Y_T by taking into account that proton and neutron states can be written as (supressing the color indices)

$$|P\rangle = |uud\rangle, \qquad |N\rangle = |udd\rangle$$

However, the weak process

$$K^+ \to e^+ + \nu_e + \pi^0 \tag{59}$$

while of course conserving Q_T, changes the eigenvalues of X_T and Y_T by ± 1, respectively. In deducing this one notices that

$$|K^+\rangle = |u\bar{s}\rangle, \qquad |\pi^0\rangle = 2^{-1/2}\{|u\bar{u}\rangle - |d\bar{d}\rangle\}$$

In strong interactions in which only hadrons (quarks) participate Q_T ($=Q_q$), X_T ($=X_q$), and Y_T ($=Y_q$) are expected to be conserved exactly. For example, the conservation of X_T is simply a consequence of isospin and charge conservation, since from general relation (33) we can now write

$$X_q = 2I_3^q - Q_q \tag{60}$$

As an illustration of the X_T conservation by strong interactions, let us analyze the reaction

$$\pi^+ + N \to K^{0,+} + \Sigma^{+,0} \tag{61}$$

Since

$$|\pi^+\rangle = |u\bar{d}\rangle, \qquad |K^0\rangle = |d\bar{s}\rangle, \qquad |\Sigma^+\rangle = |uus\rangle$$

$$|\Sigma^0\rangle = 2^{-1/2}\{[|ud\rangle + |du\rangle]|s\rangle\}$$

we see that the eigenvalues for X_q are

$$X_q(N) = X_q(K^0) = -1, \qquad X_q(\pi^+) = X_q(\Sigma^+) = 1$$

$$X_q(K^+) = X_q(\Sigma^0) = 0$$

With these one indeed verifies that the X_q charge is conserved by (61).

The electromagnetic interactions of leptons and quarks (hadrons) conserve Q_T, X_T, and Y_T. This simply follows because the electromagnetic interactions are "minimal," i.e., they are strictly proportional to the electric charges, implying that Q_T, X_T, and Y_T are the "constants of motion." By similar arguments one concludes that the neutral current interactions of leptons and quarks (hadrons) should conserve Q_T, X_T, and Y_T.

The usefulness of the dual quantum number operators was already demonstrated for four leptons, where Y_l' was identified with L_e. Furthermore, the existence of the dual electric charge quantum number, Q', led us to believe in the existence of dual electromagnetism (Šoln, 1979). Subsequently, a unified gauge theory of weak, electromagnetic, and dual electromagnetic interactions was formulated (Šoln, 1980), showing a very good agreement with experiments (Šoln, 1979, 1980).

As our $SU(2)\times U(1)$ type of algebras require the basic elementary particles to come in even numbers, we see why the GIM (Glashow, Iliopoulos, and Maiani, 1970) mechanism required at least four quarks when extending the gauge electro-weak lepton models of Weinberg (1967) and Salam (1968) to include also hadrons.

In summary we can say that $SU(2)\times U(1)$ algebras of flavor quantum numbers appear to be useful not only in the classification of various processes with respect to known elementary particle interactions but also in writing down the unified gauge theories of these interactions.

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